Inversion Parking Functions

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Parking Functions PF_n

Suppose n cars are in line to park in n spots, each car having a spot preference a_i. In order, each car i parks in the first empty spot *at or after* its preference a_i. The tuple $\alpha = (a_1, \ldots, a_n)$ is a **parking function** if all cars can park without driving past the last spot.

- A tuple α is a parking function iff there is a rearrangement α' such that $a'_i \leq i$ for all i.

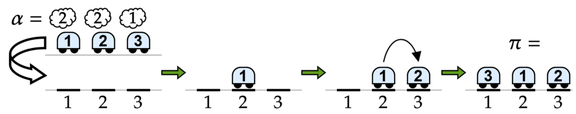


Figure: $(2, 2, 1)$ is a parking function with outcome 312.

The **outcome** $\mathcal{O} : \mathrm{PF}_n \to S_n$ of a parking function α is the permutation given by the word $\pi = \pi_1 \ldots \pi_n$ made by the cars in the road after parking.

Inversion Tables IT_n

Inversion tables are in bijection with permutations and are a useful tool to study them. An **inversion table** is a tuple $\alpha = (a_1, \ldots, a_n) \in [n]^n$ satisfying

$$a_1 \leq n, \quad a_2 \leq n-1, \quad \ldots, \quad a_n = 1$$

- Every inversion table α is a parking function (use the reverse rearrangement α'), so we can consider $\mathcal{O}(\alpha)$ when $\alpha \in \mathrm{IT}_n$, or $\mathcal{O} : \mathrm{IT}_n \to S_n$.

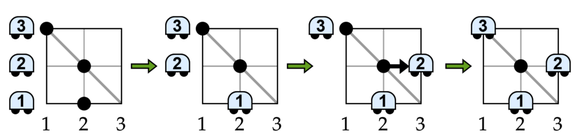


Figure: Maximal preferences (black dots) of inversion tables lie at $a_i = n - i + 1$ (the gray antidiagonal).

Goal: Enumerate $\mathcal{O}(\mathrm{IT}_n)$

Every permutation is the outcome of a parking function, but not necessarily an inversion table. We want to study and enumerate permutations that are **inversion table outcomes**. We will see that they biject to **set partitions**, which are counted by the *Bell numbers* $1, 2, 5, 15, 52, 203 \ldots$

Step 1: Characterize $\mathcal{O}(\mathrm{IT}_n) \subset S_n$

The **permutation diagram** of π is the collection of entries (i, π_i). The **arm-leg diagram** of π additionally has an antidiagonal with arms and legs coming out of entries *at or to the right* of the antidiagonal.

- $\pi \in S_n$ is an inversion table outcome iff its arm-leg diagram is **non-intersecting**.

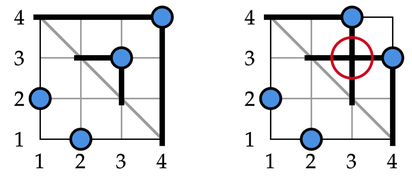


Figure: $2134 \in \mathcal{O}(\mathrm{IT}_n)$, but $2143 \notin \mathcal{O}(\mathrm{IT}_n)$.

Step 2: Define $\Phi : \mathcal{O}(\mathrm{IT}_n) \to (_)_n$

Because arm-leg diagrams of inversion table outcomes are non-intersecting, they may be assigned balanced spaced parenthesizations.

Balanced Spaced Parenthesizations $(_)_n$

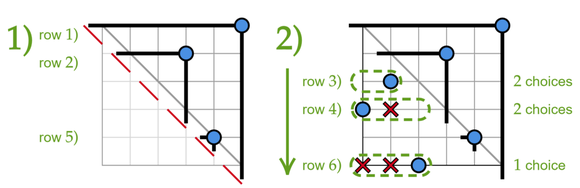
Starting with n spaces, a **spaced parenthesization** of length n may have up to 1 '(' before each space and 1 ')' after each space. It is **balanced** if

1. We can match all parentheses into pairs.
2. Every space lies in some matched pair.

So _) (_ (_ (_) and _ (_ (_) _) are not balanced.

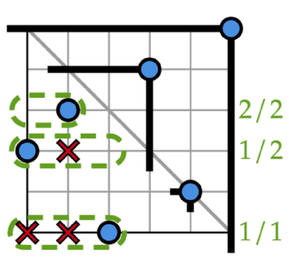
Step 3: Find $\Phi^{-1}(P)$ for $P \in (_)_n$

1. Place an entry at or to the right of each space with a '(' by undoing Step 2 (this is unique).
2. From top to bottom, place an entry to the left of each space without a '('. The number of choices equals the number of '()'s containing the space.



The preimage of a parenthesization P is parameterized by words $g_P \in G_P$ indexing the choices at each step. Thus, $\mathcal{O}(\mathrm{IT}_n)$ bijects to the set of pairs (P, g_P).

Result: $\mathcal{O}(\mathrm{IT}_n) \xleftrightarrow{\text{Steps 2-3}} (_)_n^{g_P} \xleftrightarrow{\text{Steps 4-5}} \Pi_n$



$\pi = 341526$

$\leftrightarrow$

$P = (_\ (_\ _\ _)(_)\ _)$ (1 2 3 4 5 6)

$g_P = 2\ 1\ 1$

$\leftrightarrow$

(1 (2 2 1)(3) 2)

||

$\{1, 4\}, \{2, 3, 6\}, \{5\}$

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Set Partitions Π_n

A **set partition** is a collection of nonempty disjoint *blocks* with union $[n]$, such as $\{1, 4\}, \{2, 3, 6\}, \{5\}$. Indexing the blocks by their minima, we can represent a set partition by a word $b_1 \cdots b_n$, where b_i is the block containing i. The above becomes 122132.

Step 4: Define $\Psi : \Pi_n \to (_)_n$

We obtain a balanced spaced parenthesization by placing '(' before the first of each number and ')' after the last of each number. The word 122132 becomes $(1\ (2\ 2\ 1)\ (3)\ 2)$ becomes $(_\ (_\ _\ _)\ (_)\ _)$.

Step 5: Find $\Psi^{-1}(P)$ for $P \in (_)_n$

1. Place increasing entries next to '('s (unique).

   $(_\ (_\ _\ _)\ (_)\ _) \quad \to \quad (1\ (2\ _\ _)\ (3)\ _)$

2. From left to right, place an entry in each space without a '('. The number of choices equals the number of '()'s containing the space.

   $(1\ (2\ _\ _)\ (3)\ _) \quad \to \quad (1\ (2\ x\ y)\ (3)\ z)$

   $\implies x, y \in \{1, 2\}, \quad z \in \{1, 2\} \setminus \{y\}$

The preimage of a parenthesization P is parameterized by words $g_P \in G_P$ indexing the choices at each step. Thus, Π_n bijects to the set of pairs (P, g_P). □

Future Work

- Can Step 1 be adapted to study other pattern-avoiding permutations and their diagrams?
- Balanced spaced parenthesizations are a Catalan object describing two Bell objects. What else do they describe? Where else might they be more natural than standard parenthesizations?
- Inversion table outcomes and set parititons have natural Catalan subsets. Are they related? Is there a natural choice of $g_P \in G_P$ for every P giving other Catalan subsets of $\mathcal{O}(\mathrm{IT}_n)$ and Π_n?
- Consider generalizations of parking functions. Under what conditions is $\mathcal{O}$ surjective?